On the Alleged Incommensurability of Newtonian and Relativistic Mass

Samuel C. Fletcher

University of Minnesota, Twin Cities

scfletch@umn.edu

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Abstract

In separate 1962 publications, Feyerabend and Kuhn introduced the concept of the incommensurability of scientific theories, using as an example (among others) Newtonian mechanics and the special theory of relativity. They suggested, in particular, that two theories employed incommensurable concepts of mass. Feyerabend took this incommensurability as a fatal objection to a kind of conceptual conservatism that he found in contemporaneous accounts of scientific explanation and reduction, which demanded that older theories were explained or absorbed into newer theories as special cases. In contrast, I develop here a unified technical framework for particle kinematics and collision mechanics that includes the Newtonian and relativistic versions as particular cases. The mass concept plays the same functional role for each model of the framework, and there is appropriate continuity between their realizations, as I prove that within this framework, certain Newtonian collision models are limits of relativistic collision models. Yet I argue that while one can understand in retrospect why Feyerabend and Kuhn were wrong about their specific claim about the concept of mass,

one can still uphold Feyerabend's insights about the inadequacies of views of scientific progress adhering to conceptual conservatism.

1 Introduction

One of the enduring philosophical debates about scientific change concerns the extent to which there is conceptual continuity across successive scientific theories. Continuity suggests a narrative about the rational development and progress of science; discontinuity suggests a counternarrative in which new theories can reject the central insights of past theories, denying any necessary accumulation of knowledge or understanding.

Famously, both Feyerabend (1962, 1966, 1970) and Kuhn (1962) provided historical evidence for discontinuity in the potential *incommensurability* of scientific theories. Although these philosophers' accounts of incommensurability differed in detail (Oberheim and Hoyningen-Huene, 2018, §4), they agreed broadly that the same term as used in different, successive theories about a common subject matter or phenomenon can in fact have different extensions or denote different concepts. Moreover, these extensions or concepts may be incompatible, in the sense of not being mutually inter-definable. Such terms are often said to denote incommensurable *concepts*, which is taken to be sufficient for their respective *theories* to be incommensurable.

One of the examples that Feyerabend and Kuhn gave to illustrate this thesis concerns the term *mass* in Newtonian and relativistic kinematics. As I elaborate in section 2, although both Feyerabend and Kuhn acknowledged the *formal* continuity between the two theories' mathematical *representations* of mass, they maintained that the Newtonian and relativistic *concepts* of mass were different because they in fact have incompatible properties. In light of this, Feyerabend particularly condemned philosophical theories of reduction and explanation that presuppose that successive theories encompass and thereby explain older theories as *special cases* of newer theories. As I also elaborate in section 2, followers of Feyerabend and

Kuhn on concepts of mass have drawn a range of consequences beyond this for the nature of reference (Field, 1974), emergence and scientific disunity (Rohrlich, 1988), and structural realism (Delhôtel, 2017).

However the situation appeared to the relevant actors in the history of physics, there is available—and has been, for some time—nearly all the ingredients needed to give a unified account of Newtonian and relativistic kinematics in which there is a single, functional concept of mass. Since those ingredients have not yet been combined and processed in such an account, my main goal in this essay is to do so in detail. I argue for this goal in three phases, corresponding to the three subsections of section 3.

The first phase (section 3.1) draws upon geometric formulations of both Newtonian and relativistic kinematics in the framework of four-dimensional differential geometry, with the worldlines of particles as certain (timelike) piecewise smooth one-dimensional submanifolds (Havas, 1964; Künzle, 1972; Malament, 2012). The common framework for these, sometimes called the "frame theory" (Ehlers, 2019), exhibits how Newtonian and relativistic kinematics are conceptually unified and distinguished by the value of a single scalar parameter, the *causality constant*, which effectively controls the "interaction" between the temporal and spatial degrees of freedom embodied in the shape of the null surface of the spacetime metric. In the frame theory, mass is a non-negative parameter that, when associated with a timelike worldline, specifies the degree to which the worldline departs from being a geodesic—following locally straight ("unforced") motion—when experiencing a net force. It is, in other words, what it sometimes called *inertial* mass.

That said, the kinematics of the frame theory provides only a partial account of the functional role of mass. Even though it shows how one frame-independent concept of mass plays a unified role across Newtonian and relativistic kinematics, it does not adjudicate between that concept and the frame-dependent concept of mass that Feyerabend and Kuhn had in mind, whose extensions coincide only in the Newtonian case. (Frame-(in)dependence refers to the (in)variance of the value of mass across reference frames, which I discuss in more detail in section 2.) Jammer (2000, p. 61), assessing these concepts, concludes "that the conflict between these two formalisms is ultimately the disparity between two competing views of the development of physical science." I argue that there is in fact good reason to prefer the frame-independent mass concept as the more fundamental one if one considers dynamics in addition to kinematics. For that, I proceed to the second phase (section 3.2), where I introduce the theory of collisions of point particles as developed by Ehlers et al. (1965) and Ehlers (1983), in which the conservation of energy is the fundamental law; mass plays an additional functional role in characterizing this energy. In this theory, the frame-independent (inertial) notion of mass, not the frame-dependent one, is fundamental—indeed, the latter is truly dispensable.

To complete my argument, it remains to show that not only is there a single functional concept of mass for the kinematics and collision dynamics of point particles in Newtonian and relativistic mechanics, but the manifestation of that concept varies continuously from case to case. For, one could object that a single, functional concept may exhibit some sort of *internal* discontinuity if the function were sufficiently contrived or ad hoc. To establish continuity, in the third phase (section 3.3) I adopt and extend results by Fletcher (2019) on the limit relations between general relativity and Newtonian gravitation to the case of the collision theory. I prove that every model of a wide set of cases in which the Newtonian collision theory has applied is the limit of a sequence of models of relativistic collision theory in which the speed of light is constant. Because the Newtonian and relativistic spacetimes have a common conceptual interpretation, as revealed in the first phase, the topology on collision models determining convergence encodes similarity of empirical predictions of quantities depending on durations, distances, and masses. Thus a convergent sequence of relativistic spacetimes is one in which the measurements of a Newtonian observer can be better and better approximated by those of the corresponding relativistic observers.

In section 4, I show how the account compiled in the previous section enables satisfactory rebuttals of two common objections to the commensurability of Newtonian and relativistic mass, even when it is conceived as I do here: the seemingly discrepant rest energies in the two kinematics, and the novelties in relativistic physics of mass-energy conversion and the non-additivity of mass. I also address a third objection, specific to the account of section 3, that it does not represent the historical Newtonian and relativistic theories as the latter was introduced in 1905. I concede the historical point, but emphasize that this does not entail that the account of section 3 is irrelevant. Rather, it just reveals that claims of incommensurability between scientific theories cannot rely too literally on the formulation of those theories, a position to which, I argue, Feyerabend was amenable.

After recounting the above in the concluding section 5, I discuss how even though Feyerabend was wrong (as was Kuhn) about the incommensurability of concepts of mass, my arguments vindicate Feyerabend's ultimate conclusions about the conceptual conservatism inherent in purely logical theories of scientific explanation and reduction. The frame theory, despite its unified concept of mass, affirms that the relationship between Newtonian and relativistic mechanics is not one of logical entailment (conditional or otherwise). Rather, it is only by embedding them into a common framework, supplemented with information about what can be measured approximately, that we understand how relativistic collision mechanics explains and reduces Newtonian collision mechanics.

2 Feyerabend, Kuhn, and their Followers on the Significance of the Incommensurability of Concepts of Mass

Why did Feyerabend and Kuhn hold in 1962 that the concepts of mass in Newtonian and relativistic physics were incommensurable? An elementary example will illustrate. Consider a point particle moving uniformly (in an inertial reference frame) with velocity v. What is its momentum, p, in this direction? It is a standard fact of Newtonian mechanics that the momentum is given by $p = mv$, where m is the particle's (Newtonian) mass. It is a standard fact of relativistic mechanics that the momentum is given by $p = m_0\gamma v$, where $\gamma = (1 - v^2/c^2)^{-1/2}$, c is the speed of light, and m_0 is the "rest mass." The lattermost is so named because if one assumes that the equation "$p = mv$" *defines* the momentum for both Newtonian and relativistic particle mechanics, then in the relativistic case, $m = m_0\gamma$, so that $m = m_0$ if and only if $v = 0$, i.e., the particle is at rest. The mass m is *frame-dependent* (or frame-variant) because its value is a function of the velocity v of the particle, which is defined only relative to a given reference frame; by contrast, the rest mass m_0 is *frame-independent* (or frame-invariant) because its value is the same across every reference frame. When $|v| \ll c$, $\gamma \approx 1$ (to first order in a Maclaurin expansion), hence $m \approx m_0$ and, in this approximation, the relativistic momentum formula coincides with the Newtonian one. The same goes for essentially any kinematical formula involving the relativistic γ factor.

Both Feyerabend and Kuhn have no truck with these observations: "Of course, the values obtained on measurement of the classical mass and of the relativistic mass will agree in the domains . . . in which the classical concepts were first found to be useful" (Feyerabend, 1981b, p. 81).[1]

> Apparently Newtonian dynamics has been derived from Einsteinian [dynamics], subject to a few limiting conditions. [. . .] Our [limiting] argument has, of course, explained why Newton's Laws ever seemed to work. In doing so it has justified, say, an automobile driver in acting as though he lived in a Newtonian universe. (Kuhn, 1996, pp. 101–2)

But even though both acknowledge the operational utility of such derivations, the coincidences of the resulting formulas do not demonstrate that the terms in formulas have the same meaning.

> Yet the derivation is spurious, at least to this point. . . . The variables and pa-

[1]In this section, all quotations are to Feyerabend's collected papers (Feyerabend, 1981b) and later versions of Kuhn's book for facility of reference.

> rameters that in the Einsteinian [equations] represented spatial position, time, mass, etc., still occur in the [Newtonian equations]; and they there still represent Einsteinian space, time, and mass. But the physical referents of these Einsteinian concepts are by no means identical with those of the Newtonian concepts that bear the same name. ... Unless we change the definitions of the variables in the [Newtonian equations], the statements we have derived are not Newtonian. If we do change them, we cannot properly be said to have *derived* Newton's laws, at least not in any sense of "derive" now generally recognized. ... Only at low relative velocities may the two [concepts of mass, in particular,] be measured in the same way, and even then they must not be conceived to be the same. (Kuhn, 1996, pp. 101–2)

A sufficient reason, according to Kuhn, that the term "mass" has different referents in the two theories is that it has different properties in them. Feyerabend elaborates that one of the differences is that the Newtonian unary, quantitative property of mass for a particle has in relativity theory become a binary function of the particle's relative velocity v:

> In classical, pre-relativistic physics the concept of mass ... was absolute in the sense that the mass of a system was not influenced (except perhaps causally) by its motion in the coordinate system chosen. Within relativity, however, mass has become a relational concept whose specification is incomplete without indication of the coordinate system to which spatiotemporal descriptions are all to be referred. ... [Despite the coincidence of the formulas in the low-velocity limit,] what is measured in the classical case is an *intrinsic property* of the system under consideration; what is measured in the case of relativity is a *relation* between the system and certain characteristics of [the domain of application]. (Feyerabend, 1981b, p. 81)

The fact that "mass" has a different logical *form* with more and different relata means, for Feyerabend, that "It is also impossible to define the exact classical concepts in relativistic terms or relate them with the help of an empirical generalization" (Feyerabend, 1981b, p. 81–2). Later, in 1965, Feyerabend emphasized that looking to general relativity is of no help to a logical reduction of the concept: "The classical, or absolute idea of mass ... cannot be defined within [general relativity]. ... *Not a single primitive descriptive term of [classical celestial mechanics] can be incorporated into [general relativity]*" (Feyerabend, 1981b, p. 115, emph. orig.).

Kuhn, infamously, drew very broad conclusions in 1962 from incommensurable theories—ones that use incommensurable concepts—about ontological relativity and the arrationality of choosing between those theories. In subsequent years, he restricted and qualified these conclusion, enough that there is a secondary literature on just what Kuhn thought about incommensurability across his career (Oberheim and Hoyningen-Huene, 2018, §2). By contrast, Feyerabend drew more focused but still important conclusions (Oberheim and Hoyningen-Huene, 2018, §3.1). He restricted his analysis to universal, fundamental scientific theories interpreted realistically, so that the incommensurability of their concepts logically entails incompatibility. His targets were philosophical theories of scientific explanation, reduction, and progress that presupposed that these activities are successful when newer theories, once empirically confirmed, expand the scope of and (perhaps conditionally) entail older, empirically confirmed theories (Hempel and Oppenheim, 1948; Nagel, 1949). These theories presuppose that scientific concepts, once enshrined in an established theory, become permanent in future science; any apparent change is just an extension of that concept to newer domains. Feyerabend took this *conceptual conservatism* to be incompatible with the actual history of science. He took the case of mass in mechanics, among many other examples from the physical sciences, to illustrate these philosophical theories' inadequacy and the deleterious consequences they would have on the real progress of science (Oberheim, 2005).[2]

Most of the subsequent literature commenting on this example of incommensurable con-

[2]In particular, these theories were strikingly at odds with Feyerabend's own methodological recommendations of the proliferation of incompatible theories, which he took to be the most fruitful strategy for scientific progress.

cepts of mass has affirmed Feyerabend's (and some of Kuhn's) more limited conclusions, expanding on its implications. For example:

- Field (1974) argues that we can expect the reference of scientific terms to be indeterminate; in particular, before the advent of relativity theory, "mass" did not determinately refer to m or m_0.[3]

- Rohrlich (1988) agrees that the positive, reductive relationship between successive physical theories can only be established formally or mathematically and that the concepts of the theories really are distinct at the cognitive level of the theorizing scientist. This leads to a pluralistic ontology and epistemic methodology, relativized to the level described by the theory one is using.[4]

- Delhôtel (2017) observes that the formal relationship between Newtonian and special relativistic kinematics is even deeper than that immediately suggested by the limiting relationships that Feyerabend and Kuhn consider, for both theories can be understood as special cases of a general transformation framework for kinematical quantities. However, he joins them in affirming that because of the purely operational and cognitive nature of this framework, it gives no succor to structural realists, such as Worrall (1989), who seek to vouchsafe ontological commitments in the structures and concepts retained across theory change.

- Gärdenfors and Zenker (2013) and Masterton et al. (2017) apply the theory of conceptual spaces to this example (and others) in order to describe in more detail *how* the

[3]Field cites Kuhn (1962) and, critically, Quine (1960), who draws the conclusion that theoretical terms are meaningless outside of the holistic context, or conceptual scheme, in which they are embedded. He does not cite Feyerabend (1962) on the matter, but Feyerabend held a similar holistic view of meaning for theoretical terms in his essay.

[4]Rohrlich (1988, p. 307) at times seems to disagree with Feyerabend (1962) about the "logical incompatibility" of successive theories because these theories can be related by limiting relations, which ensures (formal) logical compatibility. As one can adduce from the above discussion, this is a misunderstanding of Feyerabend, who expressly agreed with this formal compatibility. The "logical incompatibility" refers to the lack of (even conditional) entailment relations between the concepts and laws of the theories, exactly what Rohrlich (1988, p. 304) affirmed when he wrote that "the coarser theory [e.g., Newtonian mechanics] cannot be written as a *function* of the finer one [e.g., relativistic mechanics] even in its mathematical formulation."

concepts of mass are different in Newtonian and relativistic physics. Their analysis proceeds by identifying the independent and dependent kinematical dimensions (e.g., mass, length, or time) of the theories, the measurement scales on which these dimensions are measured, law-like constraints on the relations of these dimensions, and their relative cognitive importance.[5]

In contrast with these extensions, in the next section I develop a framework to exhibit the formal *and* conceptual continuity of mass across Newtonian and relativistic kinematics and collision mechanics.

3 The Common Structure of Mechanics across the Relativity Revolution

3.1 The Frame Theory

In 1981, Ehlers developed the frame theory to study systematically the relationship between Newtonian gravitation and general relativity (Ehlers, 2019). For present purposes, I will focus on the subtheory that contains just Galilean and Minkowskian spacetimes. This simplifies matters because it allows me to fix in all models the spacetime geometry, which would otherwise covary with the energy-momentum distribution according to Einstein's or Poisson's field equation. (Without the simplification, the conclusions I reach are largely the same but more complicated to express.)

Models of this subtheory of the frame theory consist of quadruples $(M, t_{ab}, s^{ab}, \nabla)$, where M is a smooth manifold of spacetime events diffeomorphic to $\mathbb{R}^4$, t_{ab} is a smooth, symmetric tensor field on M representing the temporal metric of spacetime, s^{ab} is a smooth, symmetric

[5] Gärdenfors and Zenker (2013, §4) study the example of Newtonian and special relativistic mechanics essentially as elaborated by Feyerabend and Kuhn; Masterton et al. (2017) conduct a similar study using the phase space formulation of these theories and a revised classification of types of change. The formulation dependence of these analyses prefigure some of my cautionary comments in section 5 about what can be concluded soberly from formalism alone in light of this dependence.

tensor field on M representing the spatial metric of spacetime, and ∇ is a flat, torsion-free derivative operator representing a notion of change in spacetime.[6] These fields are required to satisfy two conditions:

Compatibility $\nabla_a t_{bc} = \mathbf{0}$ and $\nabla_a s^{bc} = \mathbf{0}$. This condition ensures that the derivative operator measures the same sort of change that the temporal and spatial metrics do. (I will further discuss the interpretation of the two metrics shortly.)

Causality $t_{ab}s^{bc} = -\kappa\delta^c_a$, for some real constant κ, called the model's *causality constant*, where δ^c_a is the Kronecker delta. This condition determines the qualitative causal structure of a frame theory model. (As we shall see, $\kappa = 0$ corresponds to the Galilean case and $\kappa > 0$ to the Minkowskian case.)

The temporal and spatial metrics are so-called because they determine the durations and lengths of timelike and spacelike curves, respectively. In more detail, for any point $p \in M$ and any tangent vector $\xi^a \in T_pM$, classify ξ^a as *timelike* when $t_{ab}\xi^a\xi^b > 0$, *spacelike* when there exists a covector ξ_a at p satisfying $s^{ab}\xi_a\xi_b > 0$ and $s^{ab}\xi_a = \xi^b$, and *null* when it is neither timelike nor spacelike. A piecewise smooth, continuous curve $\gamma : I \to M$ is then said to be timelike/spacelike/null when its tangent vector ξ^a is everywhere timelike/spacelike/null. I will take on the standard assumption that timelike and spacelike curves are parameterized so that their tangent vectors have the same unit magnitude.[7] Then the duration of a timelike curve is $\int_I (t_{ab}\xi^a\xi^b)^{1/2}\,du = |I|$ and the length of a spacelike curve is $\int_I (s^{ab}\xi_a\xi_b)^{1/2}\,du = |I|$. The integrands of these expressions are called the temporal and spatial magnitudes of the vectors ξ^a, respectively. Spacelike curves represent a "path in space" (at a time). Timelike curves represent worldlines, the possible histories of massive point particles. Such

[6]See Malament (2012, Ch. 1.4) for more on the abstract index notation for differential geometry that I employ here.

[7]In section 3.3, I will appeal to results by Fletcher (2019) that drop this assumption to encode how spacetime models can flexibly represent different physical units with the same mathematical model. This plays a role in showing that a sequence of relativistic models with widening lightcones can nonetheless all represent the same (actual) speed for light. I will keep these complications otherwise tacit since they does not materially change my conclusions.

a particle has associated with its worldline a timelike vector field P^a, co-oriented with its tangent vector everywhere, called its four-momentum. The particle's mass m is defined as the temporal magnitude of its four-momentum: $m = (t_{ab}P^aP^b)^{1/2}$. Newton's first law can then be expressed as $F^b = \xi^a\nabla_a P^b$, where F^b is the net four-force on the particle. If the mass of the particle is constant—this will be a consequence of the collision theory developed in section 3.2 but is not presently required—then this can be expressed as $F^b = m\xi^a\nabla_a\xi^b$, where $\xi^a\nabla_a\xi^b$ is the four-acceleration.

Galilean spacetimes are a models of the frame theory. These are models for which t_{ab} has signature $(+, 0, 0, 0)$, s^{ab} has signature $(0, +, +, +)$, and $\kappa = 0$. Thus the tangent space at every point partitions into a single null vector, the zero vector, a 3-dimensional subspace (minus the zero element) of spacelike vectors, with the remainder as timelike vectors. The spacetime is foliated by 3-dimensional spacelike hypersurfaces, each of which represents "space at a time."

Minkowskian spacetimes are also models of the frame theory. These are models for which t_{ab} is the Minkowsi metric η_{ab}, which has Lorentz signature $(+, -, -, -)$, $s^{ab} = -\kappa\eta^{ab}$, and $\kappa = c^{-2}$, where c is the speed of light in vacuo.[8] The tangent space at every point partitions into the familiar double cone of null vectors, whose interior consists of the timelike vectors and whose exterior consists of the spacelike vectors.

Importantly, whether the frame theory model considered is Galilean or Minkowskian makes no difference to the aforementioned characterization of massive particle worldlines, their four-momenta, or Newton's first law. Some have taken the very possibility of a common four-dimensional, geometric formulation of particle kinematics to show that there is only one mass concept in play across Newtonian and relativistic kinematics, disproving incommensurability (or related) claims regarding mass (Earman and Fine, 1977; Rivadulla, 2004). Part of the argument for this is from theoretical simplicity: the invariant (frame-independent)

[8] It is also possible to allow for metrics of Lorentz signature $(-, +, +, +)$, in which case $t_{ab} = -\eta_{ab}$ and $s^{ab} = \kappa\eta^{ab}$. But adding these possibilities just complicates matters without adding any expressive power, so I exclude them by convention here.

mass concept as defined above is sufficient to formulate particle kinematics. Why introduce another mass concept, such as the frame-dependent one ($m_0\gamma$) from section 2 Okun, 1989, cf.? If needed for pragmatic reasons, it can be defined in terms of the invariant one.

I am sympathetic to this line of argument. But from the perspective of the advocate for a frame-dependent mass, it may be inconclusive. For, they could argue, one can well introduce the frame-dependent mass and then define the frame-independent ("rest") mass in terms of it, literally as the frame-dependent mass in a comoving frame. Moreover—and anticipating the objections of section 4—identifying the frame-invariant mass as the fundamental one doesn't explain why in relativistic contexts it seems to have very different properties than in Newtonian contexts. (For instance, it is non-additive for composite systems and entails a positive, rather than zero, energy for a massive body at rest.) To do that, I advise to consider the function of mass in the simplest of theories of dynamics, namely the theory of particle collisions. Although it is highly idealized, it will suffice to illustrate the dynamic function of mass.

3.2 Collisions in the Frame Theory

To consider particle collisions, we explicitly add representations of particles to the models of the frame theory. For each model, this consists in a collection of piecewise smooth, continuous, co-directed timelike curves $\overset{\alpha}{\gamma} : \overset{\alpha}{I} \to M$, an internal state space Z for the particles, and for each particle α a state-space function $\overset{\alpha}{z} : \overset{\alpha}{I} \to Z$.[9] We need not assume anything about the structure of state space.

A *collision* is a spacetime event where at least two (images of the closures of) particle worldlines intersect. The particles undergoing collisions can be classified (not mutually exclusively) into the *incoming* and *outgoing* particles: the incoming ones are those whose woldlines intersect every past neighborhood of the collision point, and the outgoing ones are

[9] For simplicity, I confine attention to massive point particles, leaving the extension to mass-0 particles (such as photons) for another occasion. Treating such particles is not a necessary part of the mass concept, and doing so carefully would involve introducing distracting details about aether-like structures in the Newtonian context.

those whose woldlines intersect every future neighborhood. We assume as well that particle worldlines are smooth in a neighborhood of any collision, so that each has a well-defined tangent vector at the collision event, and that each collision event has only finitely many incoming and outgoing particles.

The dynamics of collisions are constrained by *total energy conservation*. To make this idea precise, we first should recognize that energy is a frame-dependent notion because the speed of a particle, on which its kinetic energy depends, depends on the frame with respect to which it is described. So, let ξ^a be the timelike (unit) component of a frame at a collision event. It represents the instantaneous spacetime trajectory of an idealized measurement device not participating in the collision. At that event, ξ^a determines a relative spatial metric, a symmetric tensor $\overset{\xi}{s}_{ab}$ with the following two properties:[10]

$$\overset{\xi}{s}_{ab}\xi^b = \mathbf{0}, \tag{1}$$

$$\overset{\xi}{s}_{ab}s^{bc} = \delta^a_c - t_{ab}\xi^b\xi^c. \tag{2}$$

The tensor $\overset{\xi}{s}_{ab}s^{bc}$ at the point where it is defined is a projection operator on the tangent space at that point, its kernel given by the subspace spanned by ξ^a. The spatial metric relative to ξ^a also allows us to define the spatial magnitude of a vector ζ^a, relative to ξ^a, as $\|\zeta^a\|_\xi = (\overset{\xi}{s}_{ab}\zeta^a\zeta^b)^{1/2}$, in analogy with the temporal and spatial magnitudes of vectors given by the temporal and (non-relative) spatial metrics.

Now let Ξ_pM denote the timelike subset of the tangent space at the collision event p. We define an energy function $E : Z \times \Xi_pM \times \Xi_pM \to \mathbb{R}$, for *any* particle with worldline tangent vector v^a at p, of the following form:

$$E(z, v^a, \xi^a) = \frac{1}{2}m(z)\|v^a - \xi^a\|^2_\xi + U(z), \tag{3}$$

where $m(z)$ and $U(z)$ are scalar functions and $m(z)$ is non-zero. Here, z represents the

[10]In the relativistic case, $\overset{\xi}{s}_{ab} = \xi_a\xi_b - \eta_{ab}$; in the Newtonian case, see Malament (2012, Prop. 4.1.2).

value of the internal state at the collision event p—i.e., if γ is the worldline of the particle, $z = z(\gamma^{-1}(p))$. The first term is the kinetic energy—$\|v^a - \xi^a\|_\xi$ is just the speed of the particle relative to the frame—and the second is the internal energy. We have identified $m(z)$ as the mass associated with internal state z. In the frame theory of collisions, mass therefore is just that real function of the internal state that is (twice) the linear scale factor by which kinetic energy increases according to the square of speed.

Remarkably, Ehlers et al. (1965) and Ehlers (1983, Prop. 1) prove that *any* function with the domain and codomain of E must take the above form if it satisfies the following three conditions:[11]

Conservation For all collisions and all $\xi^a \in \Xi_p M$, where p is the collision event, $\sum_i E(\overset{i}{z}, \overset{i}{v}{}^a, \xi^a) = \sum_o E(\overset{o}{z}, \overset{o}{v}{}^a, \xi^a)$, where i indexes all incoming particles and o indexes all outgoing particles.

Continuous, Non-constant Speed Dependence $E(z, v^a, \xi^a)$ depends on v^a and ξ^a only through $\|v^a - \xi^a\|_\xi$, and is continuous and non-constant in this argument for any fixed z.

Pairwise Isotropy Consider any two particles with the same internal state z that are either both incoming or outgoing in a collision event. The sum of their energies in any frame is invariant under spatial rotations.

Speed dependence is self-explanatory; pairwise isotropy entails that any energy transferred from the pair of particles to a stationary one (such as in a completely inelastic collision) does not depend on the direction of collision. Once we have accepted conservation and the form of the energy equation (3), conservation of four-momentum follows (Ehlers et al. 1965, Thm. 1; Ehlers 1983, Prop. 2), where we identify $m(z)$ as the mass associated with internal

[11] I have adapted or modified the statements of these conditions to my notation and to facilitate their interpretation.

state z:

$$\sum_i m(\overset{i}{z})\overset{i}{v}^a = \sum_o m(\overset{o}{z})\overset{o}{v}^a. \tag{4}$$

This gives further support to the interpretation of $m(z)$ as the mass associated with the internal state z. Rather than postulating mass or four-momentum and their connection with four-force through the four-dimensional version of Newton's first law, this approach derives these connections by first defining energy and then postulating its conservation, which is and has been supported by numerous experimental results.

Importantly, this structure holds for both the Newtonian and relativistic models, and all further conclusions about conserved quantities follow derivatively from it. For instance, given a frame with timelike component ξ^a, in general the four-momentum P^a decomposes into two terms, one parallel to ξ^a and another linearly independent of it:

$$P^a = (t_{bc}\xi^b P^c)\xi^a + (s^{ab}\overset{\xi}{s}_{bc})P^c. \tag{5}$$

Conservation of four-momentum entails that each of these terms is conserved independently. The second term is defined as the three-momentum of the particle relative to the frame. The interpretation of the first term depends on the causality constant. In the Newtonian case, where $\kappa = 0$, $t_{bc}\xi^b P^c = m$, so in any collision event, the sum of the masses is conserved: $\sum_i m(\overset{i}{z}) = \sum_o m(\overset{o}{z})$. In the relativistic case, where $\kappa > 0$, $(t_{bc}\xi^b P^c) = \xi_c P^c$ is usually interpreted as the energy of the particle relative to the frame. To consider the support for this interpretation, we must turn again to our frame-dependent definition of energy, equation 3.

By straightforward calculation, this equation reduces to the following in each of the Newtonian and relativistic cases:

$$E(z, v^a, \xi^a) = \begin{cases} \frac{1}{2}m(z)\overset{\xi}{s}_{ab}v^a v^b + U(z) & \text{if } \kappa = 0, \\ \frac{1}{\kappa}m(z)(v^a\xi_a - 1) + U(z) & \text{if } \kappa > 0. \end{cases} \tag{6}$$

If we let v denote the relative speed of the particle in the frame denoted, then $\overset{\xi}{s}_{ab} v^a v^b = v^2$ when $\kappa = 0$ and $(v^a \xi_a - 1)/\kappa = c^2(1 - v^2/c^2)^{-1/2} - c^2$ when $\kappa = c^{-2}$. Thus the first term in each case represents the familiar kinetic energy.

The frame theory of collisions does not, as far as I have developed it so far, determine what the internal energy $U(z)$ is for a massive particle, except that it must be the same function of the internal state of the particle regardless of whether the model of the frame theory is Newtonian or relativistic. In particular, it does not determine Einstein's most famous equation, "$E = mc^2$" for an object at rest, which would follow from $U = mc^2$ for a structureless massive particle, nor that $\xi_a P^a$ is the total energy of such a particle relative to the frame. To do so, one must appeal to another principle. Each of the following assumptions in the relativistic case is sufficient:

1. There is a possible collision process for every massive particle to decompose into two mass-0 particles (Fernflores, 2019, §3.1).

2. $E(z, v^a, \xi^a)$ can be expressed as $E^a \xi_a$, where E^a is a tensorial function only of z and v^a (cf. Ehlers, 1983, p. 33).

3. The frame theory extends to general relativity, where Einstein's field equation constrains further the relationship between mass and total energy (Ehlers et al., 1965, p. 997).

Regardless of which assumption one adopts, the result has been experimentally verified for a variety of nuclear processes (Fernflores, 2019, §4). The resulting expression—$U = mc^2$—does not vary with κ, but depends only on the actual value of c. Thus the same expression is well-defined for both relativistic and Newtonian models of the frame theory. It also establishes that the frame-independent mass plays a functional role in quantifying the internal energy of a particle for Newtonian and relativistic models. Although one can proceed to define the frame-dependent mass from the frame-independent mass, there is no evident, additional functional role for it.

3.3 The Limit Relation Is Not Merely Formal

In section 3.1, I showed that there is a common formal and conceptual framework for Newtonian and relativistic mechanics—the frame theory—with a single, frame-independent mass concept. Although one could develop an alternative to this frame theory with a frame-dependent mass concept, the dynamical treatment of collisions in section 3.2 showed that the frame-independent concept plays a functional role in the definition of energy that the frame-dependent concept does not. It still remains to be shown, however, that this mass concept has not just a unified definition across the Newtonian and relativistic cases, but that this unification allows for continuous variation from the relativistic models to the Newtonian models. Exhibiting such conceptual continuity—not just formal continuity—is the goal of the present subsection.

To achieve this goal, I will adapt results from Fletcher (2019) on the Newtonian limit of general relativity. Although my present concern is not with gravitational theory, Fletcher (2019, §§3, 4.1, 5) shows in particular that there is a one-parameter family of Minkowskian spacetimes that converge to Galilean spacetime in the C^2 compact-open topology on models of the frame theory. This means that all scalar quantities defined on compact regions of the Minkowskian spacetimes formed by contraction of tensors with the temporal and spatial metrics and their derivatives up to second order converge to their respective values on the Galilean spacetime. Here, convergence is understood as the vanishing of the suprema of the differences of the Minkowskian and Galilean quantities over these compact regions with respect to a norm defined by any frame field. (For more details, see Fletcher (2019, §3).)

Qualitatively, convergence of this sort means that (ideally) observable quantities at point events and across compact regions in the Minkowskian spacetimes, such as durations, approximate those in the Galilean spacetime arbitrarily well as the parameter runs to its limit. Geometrically, these models keep the same underlying manifold but widen the null cones until they become hyperplanes in the limit. Conceptually, this widening can be interpreted as keeping the speed of light fixed while letting the causality constant and relative velocities of particles decrease to zero without changing the latter's worldlines. Thus in this sense the limit of the family considered keeps fixed the facts about particle trajectories and collisions while varying the spacetime structure. (For more details, see Fletcher (2019, §2).)

There is a variety of limiting results one can prove about models of the frame theory of collisions. The one that I prove below is not the most general, but it involves relatively little additional technical development and is easy to interpret. For it, I will make two background assumptions. First, I will consider only systems of structureless, massive particles with internal energy mc^2, as is standard in relativity theory and which I adopt across all models of the frame theory, per the arguments of the previous subsection.[12] Second, I need a criterion of convergence for the representation of particle states in the models. The first background assumption ensures that the internal state of a particle only enters into the energy of a particle—hence, into a model of the frame theory—through the mass function on Z. So for present purposes, instead of representing a particle α with a worldline and state function $(\overset{\alpha}{\gamma}, \overset{\alpha}{z})$, we can represent it with a worldline and a mass function, $(\overset{\alpha}{\gamma}, \overset{\alpha}{m})$, where $\overset{\alpha}{m} = m \circ \overset{\alpha}{z}$. These mass functions are just real-valued functions of a single variable, so we can say that the mass functions $\overset{\alpha}{\mu}(\lambda)$ converge to $\overset{\alpha}{m}$ as $\lambda \to 0$ just in case these functions converge pointwise in the usual topology of the real line.

Proposition. *Let $(M, t_{ab}, s^{ab}, \nabla, \{(\overset{\alpha}{\gamma}, \overset{\alpha}{m})\})$ be a Newtonian model of the frame theory of collisions satisfying the following properties:*

1. *There is a global inertial frame field according to which each particle α always has speed less than c, i.e., if ξ^a is the frame's timelike component, then $\|\overset{\alpha}{v}{}^a - \xi^a\|_\xi < c$ for all α at any point where $\overset{\alpha}{v}{}^a$ is defined.*

2. *Each $\overset{\alpha}{\gamma}$ is a piecewise geodesic of ∇.*

3. *For every model of the frame theory and every $A > 0$, there is some internal state z*

[12] It is possible to prove an analogous result without this assumption, but then one needs another assumption for the internal energy, U, analogous to the one for the mass, which is the second background assumption.

such that $m(z) = A$.

Then there is a one-parameter family of relativistic models of the frame theory of collisions, $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \{(\overset{\alpha}{\gamma}, \overset{\alpha}{\mu}(\lambda))\})$, with $\lambda \in (0,1]$, such that $\lim_{\lambda\to 0}(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \{(\overset{\alpha}{\gamma}, \overset{\alpha}{\mu}(\lambda))\}) = (M, t_{ab}, s^{ab}, \nabla, \{(\overset{\alpha}{\gamma}, \overset{\alpha}{m})\})$ in the C^2 (product) compact-open topology for the fields and point-wise for the mass functions of the particles.

Proof. We construct a one-parameter family of relativistic models of the frame theory of collisions. Then we prove that this family has the desired limiting properties.

Let $\overset{\lambda}{t}_{ab} = t_{ab} - (\lambda/c^2)\overset{\xi}{s}_{ab}$ be a Minkowskian metric on M, let $\overset{\lambda}{\nabla}$ be its Levi-Civita derivative operator, and let $\overset{\lambda}{s}{}^{ab} = -(\lambda/c^2)\overset{\lambda}{t}{}^{ab}$, so that $\overset{\lambda}{\kappa} = \lambda/c^2$. By construction, these fields satisfy the conditions for the temporal metrics, spatial metrics, and derivative operators of relativistic models of the frame theory. From the first two assumptions, it follows that the $\overset{\alpha}{\gamma}$ are each timelike piecewise geodesics of $\overset{\lambda}{\nabla}$ for each λ. So each represents a worldline of a massive particle, as required.

To complete the construction of a family of relativistic models of the frame theory of collisions, we must select mass functions $\overset{\alpha}{\mu}(\lambda)$ so that energy conservation is satisfied in every collision. Consider an arbitrary segment of an arbitrary worldline $\overset{\alpha}{\gamma}$ that is incoming or outgoing to at most one collision event. In the Newtonian model, the particle α it represents has energy $\frac{1}{2}\overset{\alpha}{m}v^2 + \overset{\alpha}{m}c^2$, where v is the particle's constant speed according to the selected inertial frame field, and its mass $\overset{\alpha}{m}$ is constant on this segment. By the third assumption, there exists a family of internal states $\overset{\alpha}{z}(\lambda)$ such that

$$\overset{\alpha}{\mu}(\lambda) = \overset{\alpha}{m}\frac{\frac{1}{2}v^2 + c^2}{\overset{\lambda}{\kappa}{}^{-1}[(1-\overset{\lambda}{\kappa}v^2)^{-1/2} - 1] + c^2}. \tag{7}$$

Assign these internal states to the segment of $\overset{\alpha}{\gamma}$ in question. Up to factors of mass, the numerator in equation 7 is the energy of the particle on this segment according to the Newtonian model and the denominator is the energy according to the relativistic model labeled by λ. Thus equation 7 guarantees that these two energies are proportional for a given λ. Because the Newtonian model satisfies conservation of energy at collision events by assumption, the relativistic models must therefore do so as well in the inertial frame selected. Hence they must satisfy energy conservation in general, since in relativistic models, conservation of energy is invariant across frames of reference (even if the specific values of energy are not). Thus we have shown that the $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}^{ab}, \overset{\lambda}{\nabla}, \{(\overset{\alpha}{\gamma}, \overset{\alpha}{\mu})\})$ are indeed relativistic models of the frame theory of collisions.

Fletcher (2019) already proved that $\lim_{\lambda \to 0}(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}^{ab}, \overset{\lambda}{\nabla}) = (M, t_{ab}, s^{ab}, \nabla)$ in the C^2 (product) compact-open topology, so it remains to be shown that $\lim_{\lambda \to 0} \overset{\alpha}{\mu}(\lambda) = \overset{\alpha}{m}$ for each α. The numerator of equation 7 is constant with respect to λ, while the denominator depends on λ through $\overset{\lambda}{\kappa} = \lambda / c^2$. Through application of L'Hôpital's rule, one can calculate that the denominator approaches $\frac{1}{2}v^2 + c^2$ as $\lambda \to 0$, which proves that $\lim_{\lambda \to 0} \overset{\alpha}{\mu}(\lambda) = \overset{\alpha}{m}$ for each α. □

Each of the proposition's assumptions plays an important role in the proof. The first enables the curves $\overset{\alpha}{\gamma}$, without any changes, to be timelike in the relativistic models. The second ensures that each curve segment has a constant velocity, so that equation 7 can assign a constant λ-dependent mass to it in the relativistic models. The third is sufficient to guarantee that the mass that equation 7 prescribes is possible. The proposition's conclusion demonstrates a kind of limiting reduction of certain models of the Newtonian frame theory of collisions by models of the relativistic frame theory of collisions (cf. Fletcher, 2019). This is not a merely formal notion of continuity, for each of the mathematical structures so demonstrated to vary continuously has a unified conceptual interpretation.

4 Three Objections Addressed

In this section, I briefly recount and rebut three objections to the commensurability of the Newtonian and relativistic concepts of mass, even when that concept is the frame-independent one considered above. The first two objections exhibit properties on which the concepts purportedly differ enough to conclude that they cannot be concepts that are either the same or continuously related. The third objection concerns the historical aptness of the frame theory to address the question of incommensurability. Each rebuttal elicits a deeper investigation of the significance of the frame theory for collisions as a unification—or "dialectical ascent" (Koertge, 1970)—of the Newtonian and relativistic mechanics of collisions.

4.1 Energy at Rest

When a massive particle is at rest in Newtonian mechanics, one traditionally ascribes it an energy of zero; when a massive particle is at rest in relativistic mechanics, one traditionally ascribes it an energy of mc^2. Since the latter cannot approximate the former in general, either the Newtonian and relativistic concepts of mass are in fact distinct, or at least the realization of the concept varies discontinuously from relativistic to Newtonian spacetimes.[13]

From the point of view of the frame theory for collisions, this objection implicitly chooses an internal energy for Newtonian models that is inconsistent with that for relativistic models. Recall that the energy functional E for a particle whose worldline has tangent vector v^a is $E(\xi) = \frac{1}{2}m(z)\|v^a - \xi^a\|_\xi^2 + U(z)$. There were arguably compelling theoretical and experimental reasons to let $U(z) = m(z)c^2$ for an unstructured massive particle in relativistic collision theory, so if one adopts this suggestion—as the premise of the objection does—then one should adopt this suggestion also for Newtonian collision theory, rather than $U(z) = 0$. This equalizes the energy of a massive particle at rest in any model of the frame theory and restores continuity to this expression.

It might be objected that there is something odd about about adopting in a Newtonian model an expression for energy like mc^2, as it refers to the speed of light, a decidedly relativistic constant and concept (Delhôtel, 2018, p. 1575). But in the frame theory, the speed of light is conceptually separate from the causality constant κ that controls the pitch of the relativistic null cones. There is no incompatibility between Newtonian kinematics

[13] Delhôtel (2018, pp. 1575–6) argues for the former option.

and a velocity scale set by c. Although I have not here described any Newtonian theory of electromagnetism—presumably, some sort of aether theory—such a theory would connect this scale to light is much the way contemporary Maxwell theory does. Setting $U(z) = m(z)c^2$ for Newtonian models moreover engenders no empirical conflict with the domain of successful application of the theory—conservation of mass for these models will entail conservation of internal energy. It also reminds us that in Newtonian mechanics, energy has often been regarded as an interval-scale quantity, rather than a ratio-scale quantity as it becomes in (general) relativistic mechanics. This shows that the measurement scale for energy, despite being a single, functionally defined quantity in the frame theory, is a relation between that quantity and the background spacetime structure, just as was the case for mass conservation in collisions and other seemingly different properties that mass manifests in the Newtonian and relativistic cases.

4.2 Mass-Energy Conversion and the Conservation and Additivity of Mass

Kuhn (1996, p. 102) pithily articulated one of the classic objections to commensurability:[14] "Newtonian mass is conserved; Einsteinian [relativistic mass] is convertible with energy." The objection thus has two connected parts. First, we saw above that in Newtonian collision models, mass was conserved in the sense that the sum of the masses of the incoming particles equaled the sum of the masses of the outgoing particles. This conservation law does not hold for relativistic models. The sum of the masses of the particles in a relativistic collision is nevertheless constrained—and this is the second part—because of its contributions both to the kinetic and internal energies of each particle. Although there is some controversy regarding in what (metaphysical) sense mass is truly "convertible with energy" (Fernflores, 2019, §2), the essential point seems to be that in relativistic collision processes, any surplus

[14]See also Torretti (1990, p. 67), Torretti (1999, p. 288), and Delhôtel (2018, p. 1576) for similar, more elaborated versions that connect also with the ideas of the non-additivity of mass in relativity theory, mentioned by Feyerabend (1981a, pp. 59–61) and discussed in footnote 16.

or deficit in the sum of the masses of the outgoing particles compared with that of the incoming particles has a corresponding proportional deficit or surplus in the sum of the kinetic energies.[15]

Electron-positron annihilation, $e^- + e^+ \to \gamma + \gamma$, is a striking example of this. All the particles involved are fundamental, hence have no internal degrees of freedom (as far as we presently understand). Each of the incoming particles has (the same) positive mass, while each of the outgoing particles has zero mass but the same total energy. Hence the internal energy associated with the mass of the incoming particles is "converted" to kinetic energy associated with the outgoing particles. The sort of "conversion" is not possible in Newtonian models, in which mass—hence, any internal energy associated with mass—is conserved in a collision process.

These observations are essentially correct, but one can resist drawing the conclusion from them that the mass concepts in Newtonian and relativistic mechanics must be incommensurable. The frame theory of collisions—which, again, has a single, functional concept of mass—explains why in different spacetime settings mass may satisfy different properties. It is because these properties *derive* from relations between mass, energy, and spacetime structure. It assumes as its only fundamental law for collisions the conservation of energy, from which conservation of four-momentum follows. What further consequences this has depends upon the causality constant in the models in question. Moreover, our limiting results of section 3.3 show that when the relative speeds between the incoming and outgoing particles in collisions are sufficiently small (for a given number of particles), conservation of mass is approximately recovered. It is only for relatively high such speeds, as are exhibited in the electron-positron annihilation example above, that any appreciable mass-energy "conversion" becomes possible.[16]

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[15]The proportional deficit or surplus may also appear in the internal energy of the outgoing particles, although in most cases of interest this energetic state is not stable, resulting in a further decay to a stable state with lower internal but higher kinetic energy, e.g., by photon emission. Examples of this sort include neutron capture by sulfur, $\mathrm{n} + {}^{32}\mathrm{S} \longrightarrow {}^{33}\mathrm{S} \longrightarrow {}^{33}\mathrm{S} + \gamma$, or by silicon, $\mathrm{n} + {}^{28}\mathrm{Si} \longrightarrow {}^{29}\mathrm{Si} \longrightarrow {}^{29}\mathrm{Si} + \gamma$ (Fernflores, 2019, §4).

[16]There is another response available to the objection of this section, that the conservation of mass in

An analogy may be helpful here. In classical physics, if gravitation is the only force that acts at a distance, then it is a universal fact about small massive bodies near the surface of the Earth that, without direct contact support, they fall to the ground. But if electrostatic forces are also present—Coulomb's constant takes on a positive value—then this universal fact no longer holds, for in some cases electrostatic repulsion can balance gravitational attraction. Whether a body falls depends on the interaction between its mass, electrical charge, and other nearby bodies' properties. The introduction of electrostatics into terrestrial mechanics does not thereby change the concepts of mass or gravitation, for these concepts still play the same functional role they did as before, even though there are different properties that mass entails. This is because many of those properties really depend on relations that mass (and gravitation) have with other physical properties. Similarly, many of the properties of mass in mechanics depend on its relations with energy and spacetime structure. No conceptual incompatibility arises when these properties vary continuously as spacetime structure varies continuously.

4.3 Historical versus Retrospective Commensurability

There is a third objection worth considering:[17] The mathematics and formulation of the four-dimensional frame theory for collisions came much after the advent of relativistic mechanics, so it cannot reflect the conceptual intent of physicists at the advent of relativity theory. But the claim about the incommensurability of concepts of mass is after all historical, one about

Newtonian collisions should really be understood as a consequence of the *equality* of mass in any collision of the *composite* incoming and outgoing particle systems, respectively. Each composite system has a four-momentum equal to the sum of the four-momenta of its components, so the mentioned equality follows immediately from the conservation of four-momentum for any collision in a model of the frame theory. For Newtonian models, equality of composite mass is equivalent to conservation of mass, but not in relativistic models. (For instance, the two photons in the electron-positron annihilation considered above have a positive composite mass exactly equal to that of the composite mass of the electron and positron.) However, this response seems just to trade one essential difference between the Newtonian and relativistic concepts of mass, non-conservation, for another, non-additivity. The best response to non-additivity is the same as for non-conservation—namely, to focus on the functional role of mass and the recovery of approximate additivity for sufficiently small relative speeds in collisions.

[17] This objection, and some aspects of my response, are modeled on "Objection 2" considered in Earman and Fine (1977). As is evident below, my response goes beyond by connecting it with Feyerabend's positive views on the relations between history, formalism, and commensurability.

the transition between and comparison of concepts as the science was in development. Thus the retrospective commensurability of concepts of mass evinced by the frame theory does not bear upon the historical incommensurability of concepts of mass in the early twentieth century.

I concede, of course, that the formalism of section 3 is not the formalism of the historical actors in question. But this does not entail that their *concepts* are unrelated. I aver that the account of Newtonian collision mechanics within section 3 is theoretically equivalent to, or at most a conservative extension of, mechanics as it was formulated at the advent of relativity theory, and likewise for the account of relativistic collision mechanics. That equivalence (or conservative extension), once embedded in the frame theory, establishes a logical connection between the theories' respective concepts.

One of the interesting conclusions to draw from this is that the usual conceptual (rather than historical) understanding of incommensurability is likely too tied to the contingent and accidental features of the particular language or formalism in which a theory may be described—that is, it is too tied to the syntactic conception of theories that dominated philosophy of science in the 1940s–60s. While there continues to be debate about the merits of the semantic view of theories, the syntactic view's successor, almost all seem to be in agreement that capturing the structure of a theory involves in large part aspects that are invariant (or at least appropriately covariant) across choice of language and formalism. In light of the new understanding afforded by the frame theory, the alleged essential differences in the case of concepts of mass do not appear to be so invariant. The single, functional concept of mass described in the frame theory is so invariant, and the other derivative mass concepts are defined in terms of it.

There is clear evidence that Feyerabend, too, maintained that mere difference in formalism does not entail conceptual, hence ontological differences. In essays on Niels Bohr's philosophy of physics in 1968 and 1969, Feyerabend emphasized the utility of Bohr's preference for qualitative, non-formal treatments of basic physical principles, for "the formal features of a theory do not always adequately mirror the physical situation" (Feyerabend, 1981b, p. 275). In a footnote to this passage, he provides an example: "This becomes clear at once when one considers how many different formalisms can be used to express the facts of classical celestial mechanics and how careful one must therefore be in the attempt to use the features of some particular formalism as a basis for ontological inference" (Feyerabend, 1981b, p. 275n66). At this point, Feyerabend quotes the discussion of Havas (1964, p. 963), who is reflecting on his own work showing that Newtonian and relativistic mechanics, including gravitation, can be given a common, four-dimensional formulation:

> The variety of ways, in part based on entirely different sets of concepts, in which we can express the fundamental laws for gravitating matter, all leading to identical physical predictions, should caution us not to put undue stress on the supposed implications of a particular formulation of a theory, *even if other formulations might not be available at a given time.* (Feyerabend's emphasis)

The "different sets of concepts" here are just the different sets of *fundamental* concepts with which each formalism begins, not theoretically inequivalent sets. Feyerabend's added emphasis underlines that the historical contingency of a particular formalism, and the concepts it suggests, should not be taken as the definitive suggestion for what the theory's fundamental concepts are.[18]

5 Conclusion: Feyerabend's Vindication

In section 3, I exhibited the frame theory of collisions, in which there is a single, functional mass concept for particles. Its function is the role it plays in the energy of a particle and

[18] I am not aware of passages where Feyerabend applies this affirmed insight to his previously discussed case of concepts of mass, but it stands to reason that if he were to have, then he would have changed his mind to come to the same conclusion as I have. But why didn't he apply them? While I suspect that there are many complex reasons for this, an important one is that section 3.3's interpretation of the limiting relation between relativistic and non-relativistic theories preserving the speed of light (c) instead of taking $c \to \infty$ is first publicly available only recently (Fletcher, 2019). This interpretation is necessary to argue that the limiting relation, hence the continuity it establishes, is not merely formal, which in turn is essential to the response to the objections in section 4.

its dynamics. In this theory, mass does not have different intrinsic properties from one model to another; rather, what properties it always a relation between mass's functional role and the ambient spacetime structure of temporal and spatial metrics. Moreover, I proved a proposition showing that a large class of Newtonian models are limits of relativistic models, meaning that, in the latter, the metrics and the masses of the particles transition continuously to those in the respective limiting Newtonian models.

Although this shows that Feyerabend was wrong about this specific alleged case of incommensurability (as was Kuhn), it does not show that he was wrong in his critique of philosophical theories of explanation and reduction. Those critiques only take incommensurability as a sufficient but not necessary condition for inadequacy. Recall from section 2 that Feyerabend's concern with incommensurability was only instrumental towards this ultimate critical goal. Thus, his conclusions about the inadequacies of these theories that take explanation and reduction to be purely logical relations could well be right for other reasons. Indeed, the structure of the frame theory of collisions *supports* this conclusion, for it exhibits in much more detail how the Newtonian models (with vanishing causality constant) are not special cases of the relativistic models (with positive causality constant). One cannot understand how the relativistic models explain the Newtonian models through a process of restriction, redefinition of terms, and entailment, because one cannot obtain the Newtonian models through such a process. Rather, one must lift to the more encompassing vantage point of the frame theory, which involves a relaxation or generalization, rather than a restriction, of the relativistic theory. It is only from that unifying framework that one can deduce the relationships and connections between the two sets of models. Far from an embrace of conceptual conservatism, the explanation and reduction afforded by the frame theory shows that re-conceiving the conceptual order between older and newer theories can resolve apparent incommensurability (cf. Koertge, 1970).

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